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Equations of Motion of a Hinged Body Over a Spherical Earth

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FOREWORD

This report describes the derivation of the equations of motion of a hinged vehicle. This work was accomplished at the Naval Air Warfare Center Weapons Division, China Lake, during the period from May 1990 to May 1992 and was funded by the Air Launched Weaponry Block, NW1A, sponsored by the Office of Naval Technology.

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The six-degree-of-freedom (6-DOF) equations of motion are derived in vector form for a hinged vehicle flying over a spherical Earth. Initially, the most general form of the equations is developed assuming that the combustor may roll with respect to the body, as well as deflect in pitch and yaw. Next, the primary case is developed in which the combustor cannot roll with respect to the body. Finally, these equations are specialized to three simpler cases. The motivation for deriving this set of equations is to support feasibility studies of the low drag ramjet (LDR), which is streamlined by deleting all aerodynamic surfaces. All directional control comes from thrust vectoring, which is accomplished by deflecting the entire motor rather than just the exhaust nozzle, as in more conventional designs. Since the motor is a large fraction of the total vehicle mass, its inertia has a significant effect on the dynamics of the overall vehicle. Sufficient detail is included to serve as a tutorial.

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The author appreciates the help of Allen Robins, who traced through this development and exercised the resulting equations in his 6-degree-of-freedom (DOF) simulation of the low drag ramjet (LDR) missile. Thanks also to Richard Boss for his independent derivation of the planar form of these equations, which served as helpful check of this work.

INTRODUCTION

The six-degree-of-freedom (6-DOF) equations of motion are derived in vector form for a hinged vehicle flying over a spherical Earth. Initially, the most general form of the equations is developed assuming the combustor may roll with respect to the body, as well as deflect in pitch and yaw. Next, the primary case is developed in which the combustor cannot roll with respect to the body. Finally, these equations are specialized to three simpler cases. The motivation for deriving this set of equations is to support feasibility studies of the low drag ramjet (LDR), which is streamlined by deleting all aerodynamic surfaces. All directional control comes from thrust vectoring, which is accomplished by deflecting the entire motor rather than just the exhaust nozzle, as in more conventional designs. Since the motor is a large fraction of the total vehicle mass, its inertia has a significant effect on the dynamics of the overall vehicle.

GENERAL SIX-DEGREE-OF-FREEDOM MOTION OVER SPHERICAL EARTH WITH ROLLING HINGE

To fully describe the trajectories of two rigid bodies, 24 first-order scalar differential equations are needed. For each body there are 12 equations, three for each of the following: position, attitude, velocity, and angular rate. In this case, however, the task

of developing the angular and translational acceleration equations is simplified by the fact that the combustor is attached to the body. The hinge makes the velocity and acceleration of the combustor's center of gravity (CG) a function of the combustor's attitude, angular velocity, and the body states. In this section, the combustor is allowed to roll, pitch, and yaw with respect to the body.

An additional simplification results from the assumption that the mass properties of each body change slowly, $\dot{m}_B \approx 0$, $\dot{m}_C \approx 0$, $\dot{\bar{\beta}}_B\Big|_B \approx 0$, $\dot{\bar{\beta}}_C\Big|_C \approx 0$, $\dot{I}_B\Big|_B \approx 0$ and $\dot{I}_C\Big|_C \approx 0$. The missile geometry is illustrated in Figure 1, and the notation is explained in detail in the notation section at the end of this report.

NEWTON'S SECOND LAW APPLIED TO WHOLE VEHICLE

Apply Newton's second law to the whole vehicle, where the forces and moments in the hinge are as illustrated in Figure 2. (Reference 1 has a good statement of Newton's second law.) External forces and moments are assumed to act at the combustor and body CGs.

$$\frac{d(m_V \bar{v}_V)}{dt} = m_V \bar{a}_V = \bar{F}_B + \bar{F}_C + m_V \bar{G} \tag{1}$$

In Equation 1, the subscript "V" refers to a quantity that applies to the overall vehicle. A subscript "C" refers to the combustor or motor section, and the "B" subscript refers to the main body.

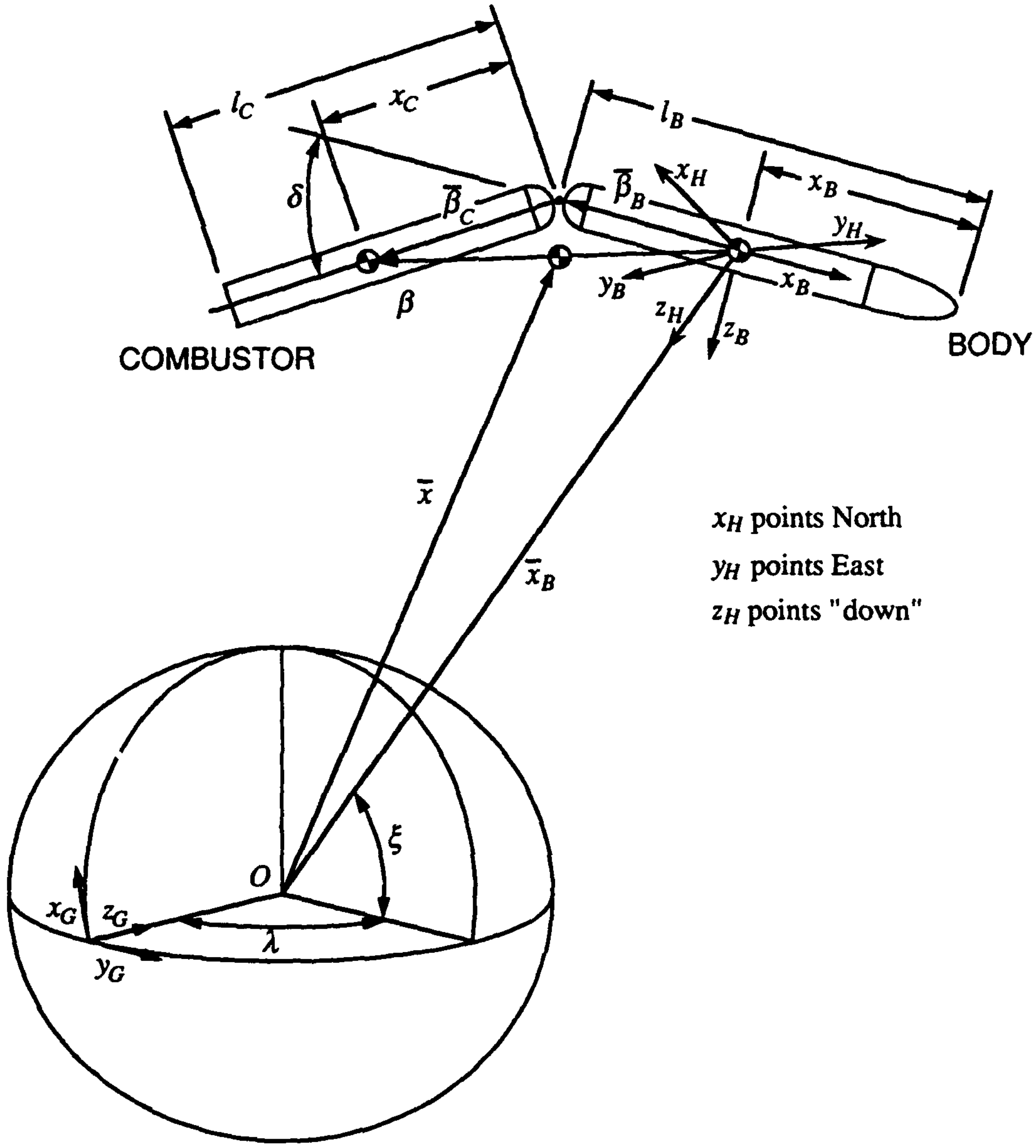

FIGURE 1. Vehicle Over Spherical Earth.

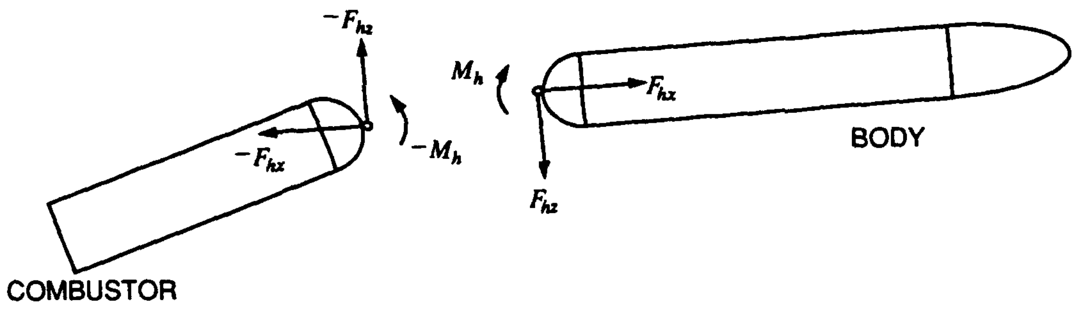

FIGURE 2. Forces and Moments in the Hinge.

Since the location of the vehicle CG relative to both bodies varies with combustor deflection angle, and the inertial measurement unit will be located in the main body, the body CG is the reference point for these equations. The body acceleration is derived from the vehicle acceleration as follows:

$$m_V\bar{a}_V = m_B\bar{a}_B + m_C\bar{a}_C$$

$$m_V\bar{a}_V = m_B\bar{a}_B + m_C\bar{a}_B + m_C\bar{a}_C - m_C\bar{a}_B$$

$$m_V\bar{a}_V = (m_B + m_C)\bar{a}_B + m_C(\bar{a}_C - \bar{a}_B)$$

$$m_V\bar{a}_V = m_V\bar{a}_B + m_C\bar{a}_{C/B}$$

$$m_V\bar{a}_B = m_V\bar{a}_V - m_C\bar{a}_{C/B} \tag{2}$$

The "$\bar{a}_{C/B}$" is the acceleration of the combustor with respect to the body. This is a noninertial acceleration. Since the combustor is attached to the main body, its position relative to the main body can be described algebraically.

$$\bar{x}_C = \bar{x}_B + \bar{x}_{C/B}$$

$$\bar{x}_C = \bar{x}_B + \bar{x}_{h/B} + \bar{x}_{C/h}$$
$$\bar{x}_C = \bar{x}_B + \bar{\beta}_B + \bar{\beta}_C$$

Differentiate for combustor velocity:

$$\bar{v}_C = \frac{d\bar{x}_C}{dt} = \bar{v}_B + \bar{v}_{C/B}$$
$$\bar{v}_C = \bar{v}_B + \left(\dot{\bar{\beta}}_B + \bar{\omega}_B \times \bar{\beta}_B\right) + \left(\dot{\bar{\beta}}_C + \bar{\omega}_C \times \bar{\beta}_C\right)$$

Remember, $\dot{\bar{\beta}}_B = 0$ and $\dot{\bar{\beta}}_C = 0$

$$\bar{v}_C = \bar{v}_B + \bar{\omega}_B \times \bar{\beta}_B + \bar{\omega}_C \times \bar{\beta}_C$$

Differentiate a second time for combustor acceleration:

$$\bar{a}_C = \frac{d\bar{v}_C}{dt} = \bar{a}_B + \bar{a}_{C/B}$$
$$\bar{a}_C = \bar{a}_B + \left[\dot{\bar{\omega}}_B \times \bar{\beta}_B + \bar{\omega}_B \times \dot{\bar{\beta}}_B + \bar{\omega}_B \times \left(\bar{\omega}_B \times \bar{\beta}_B\right)\right]$$
$$+ \left[\dot{\bar{\omega}}_C \times \bar{\beta}_C + \bar{\omega}_C \times \dot{\bar{\beta}}_C + \bar{\omega}_C \times \left(\bar{\omega}_C \times \bar{\beta}_C\right)\right]$$

Since $\dot{\bar{\beta}}_B$ and $\dot{\bar{\beta}}_C$ are assumed to be near zero,

$$\bar{a}_C = \bar{a}_B + \dot{\bar{\omega}}_B \times \bar{\beta}_B + \bar{\omega}_B \times \left(\bar{\omega}_B \times \bar{\beta}_B\right) + \dot{\bar{\omega}}_C \times \bar{\beta}_C + \bar{\omega}_C \times \left(\bar{\omega}_C \times \bar{\beta}_C\right)$$

Isolating the acceleration of the combustor with respect to the body,

$$\bar{a}_{C/B} = \dot{\bar{\omega}}_B \times \bar{\beta}_B + \bar{\omega}_B \times \left(\bar{\omega}_B \times \bar{\beta}_B\right) + \dot{\bar{\omega}}_C \times \bar{\beta}_C + \bar{\omega}_C \times \left(\bar{\omega}_C \times \bar{\beta}_C\right) \tag{3}$$

Inserting Equations 1 and 3 into Equation 2 yields an expression for the inertial acceleration of the main body CG

$$m_V \bar{a}_B = \bar{F}_B + \bar{F}_C + m_V \bar{G}$$
$$-m_C\left[\dot{\bar{\omega}}_B \times \bar{\beta}_B + \bar{\omega}_B \times \left(\bar{\omega}_B \times \bar{\beta}_B\right) + \dot{\bar{\omega}}_C \times \bar{\beta}_C + \bar{\omega}_C \times \left(\bar{\omega}_C \times \bar{\beta}_C\right)\right] \tag{4}$$

Because the body and combustor angular rates are relative to inertial space, they include the rates attributable to the spherical, rotating Earth.

$$\bar{\omega}_C = \bar{\omega}_{C/B} + \bar{\omega}_{B/H} + \bar{\omega}_{H/G} + \bar{\omega}_G$$

$$\bar{\omega}_B = \bar{\omega}_{B/H} + \bar{\omega}_{H/G} + \bar{\omega}_G$$

The inertial acceleration of the body CG is now known. But it is more useful to know the body acceleration relative to the body frame, $\dot{\bar{v}}_{B/G}\big|_G$ (see the notation section), since this is what a strapdown accelerometer located at the body CG would measure. Expanding the body CG acceleration in terms of body angular velocity yields

$$\bar{a}_B = \frac{d^2\bar{x}_B}{dt^2} = \frac{d}{dt}\left(\dot{\bar{x}}_B\big|_G + \bar{\omega}_G \times \bar{x}_B\right)$$

$$\bar{a}_B = \ddot{\bar{x}}_B\big|_G + 2\bar{\omega}_G \times \dot{\bar{x}}_B\big|_G + \dot{\bar{\omega}}_G \times \bar{x}_B + \bar{\omega}_G \times (\bar{\omega}_G \times \bar{x}_B)$$

$$\bar{a}_B = \dot{\bar{v}}_{B/G}\big|_G + 2\bar{\omega}_G \times \bar{v}_{B/G} + \bar{\omega}_G \times (\bar{\omega}_G \times \bar{x}_B)$$

where

$$\dot{\bar{v}}_{B/G}\big|_G = \dot{\bar{v}}_{B/G}\big|_B + \bar{\omega}_{B/G} \times \bar{v}_{B/G}$$

and $\dot{\bar{\omega}}_G$ is assumed to be zero.

$$\bar{a}_B = \dot{\bar{v}}_{B/G}\big|_B + (\bar{\omega}_{B/H} + \bar{\omega}_{H/G}) \times \bar{v}_{B/G} + 2\bar{\omega}_G \times \bar{v}_{B/G} + \bar{\omega}_G \times (\bar{\omega}_G \times \bar{x}_B)$$

Rearrange $\bar{a}_B$ to isolate the different types of acceleration.

$$
\begin{array}{llll}
\bar{a}_B = \dot{\bar{v}}_{B/G}\big|_B + \bar{\omega}_{B/H} \times \bar{v}_{B/G} & & & \text{Flat Earth} \\
\quad +\bar{\omega}_{H/G} \times \bar{v}_{B/G} & & & \text{Spherical Earth} \\
\quad +2\bar{\omega}_G \times \bar{v}_{B/G} & \text{Coriolis} & \left.\begin{matrix} \\ \\ \end{matrix}\right\} & \text{Rotating Earth} \\
\quad +\bar{\omega}_G \times \left(\bar{\omega}_G \times \bar{x}_B\right) & \text{Centrifugal relief} & &
\end{array}
$$

Inserting this into Equation 4 and placing all derivatives on the left side yields the first of the three vector differential equations for translational and angular velocities.

$$
\begin{aligned}
m_V \dot{\bar{v}}_{B/G}\big|_B &+ m_C\left(\dot{\bar{\omega}}_C \times \bar{\beta}_C + \dot{\bar{\omega}}_B \times \bar{\beta}_B\right) \\
&= \bar{F}_B + \bar{F}_C \times m_V \bar{G} - m_C\left[\bar{\omega}_C \times \left(\bar{\omega}_C \times \bar{\beta}_C\right) + \bar{\omega}_B \times \left(\bar{\omega}_B \times \bar{\beta}_B\right)\right] \\
&\quad - m_V\left[\bar{\omega}_{B/G} \times \bar{v}_{B/G} + 2\bar{\omega}_G \times \bar{v}_{B/G} + \bar{\omega}_G \times \left(\bar{\omega}_G \times \bar{x}_B\right)\right]
\end{aligned}
\tag{5}
$$

CONSERVATION OF ANGULAR MOMENTUM FOR MAIN BODY

Conservation of angular momentum for the main body about its own CG is the starting point for the next derivation.

$$\frac{d}{dt}\left(\tilde{I}_B \bar{\omega}_B\right) = \bar{M}_B + \bar{M}_h + \bar{\beta}_B \times \bar{F}_h$$

At this point, an assumption is made about the forces internal to the hinge. The servo is assumed to exert a pure couple (see Figure 2). Thus, the force internal to the hinge between the body and the combustor is defined by Newton's third law. (Reference 1 has a good statement of Newton's third law.) This force is not generally measured, so it is eliminated by applying Newton's second law to the combustor, solving for the force in the hinge and substituting for it in the angular momentum equation for the body.

$$m_C \bar{a}_C = -\bar{F}_h + \bar{F}_C + m_C \bar{G}$$

Substitute for $\overline{F}_h$,

$$\frac{d}{dt}\left(\tilde{I}_B\overline{\omega}_B\right)=\overline{M}_B+\overline{M}_h+\overline{\beta}_B\times\left(\overline{F}_C+m_C\overline{G}-m_C\overline{a}_C\right)$$

Expan˙ out the derivative on the left side.

$$\dot{\tilde{I}}_B\Big|_B\overline{\omega}_B+\tilde{I}_B\dot{\overline{\omega}}_B+\overline{\omega}_B\times\tilde{I}_B\overline{\omega}_B=\overline{M}_B+\overline{M}_h+\overline{\beta}_B\times\left(\overline{F}_C+m_C\overline{G}-m_C\overline{a}_C\right)$$

Remember, $\dot{\tilde{I}}_B\Big|_B\approx 0$, expand out and place all derivatives on the left hand side:

$$\begin{aligned}\tilde{I}_B\dot{\overline{\omega}}_B&+\overline{\beta}_B\times m_C\left(\dot{\overline{v}}_{B/G}\big|_B+\dot{\overline{\omega}}_C\times\overline{\beta}_C+\dot{\overline{\omega}}_B\times\overline{\beta}_B\right)\\&=\overline{M}_B+\overline{M}_h+\overline{\beta}_B\times\left(\overline{F}_C+m_C\overline{G}\right)-\overline{\omega}_B\times\tilde{I}_B\overline{\omega}_B\\&-\overline{\beta}_B\times m_C\left[\overline{\omega}_{B/G}\times\overline{v}_{B/G}+2\overline{\omega}_G\times\overline{v}_{B/G}+\overline{\omega}_G\times\left(\overline{\omega}_G\times\overline{x}_B\right)\right]\\&-\overline{\beta}_B\times m_C\left[\overline{\omega}_C\times\left(\overline{\omega}_C\times\overline{\beta}_C\right)+\overline{\omega}_B\times\left(\overline{\omega}_B\times\overline{\beta}_B\right)\right]\end{aligned}\qquad(6)$$

CONSERVATION OF ANGULAR MOMENTUM FOR COMBUSTOR

The development of the angular momentum equation of the combustor about its own CG is very similar to that for the body. At this point, the combustor is assumed to be able to pitch, yaw, and roll with respect to the body.

$$\begin{aligned}\frac{d}{dt}\left(\tilde{I}_C\overline{\omega}_C\right)&=\overline{M}_C-\overline{M}_h-\overline{\beta}_C\times(-\overline{F}_h)\\&=\overline{M}_C-\overline{M}_h+\overline{\beta}_C\times(\overline{F}_C+m_C\overline{G}-m_C\overline{a}_C)\end{aligned}$$

Expand out, assuming $\dot{\tilde{I}}_C\big|_C \approx 0$ and rearrange.

$$\begin{aligned}\tilde{I}_C\dot{\bar{\omega}}_C + \bar{\beta}_C \times m_C\left(\dot{\bar{v}}_{B/G}\big|_B + \dot{\bar{\omega}}_C \times \bar{\beta}_C + \dot{\bar{\omega}}_B \times \bar{\beta}_B\right)& \\ = \bar{M}_C - \bar{M}_h + \bar{\beta}_C \times \left(\bar{F}_C + m_C\bar{G}\right) - \bar{\omega}_C \times \tilde{I}_C\bar{\omega}_C& \\ -\bar{\beta}_C \times m_C\left[\bar{\omega}_{B/G} \times \bar{v}_{B/G} + 2\bar{\omega}_G \times \bar{v}_{B/G} + \bar{\omega}_G \times \left(\bar{\omega}_G \times \bar{x}_B\right)\right]& \\ -\bar{\beta}_C \times m_C\left[\bar{\omega}_C \times \left(\bar{\omega}_C \times \bar{\beta}_C\right) + \bar{\omega}_B \times \left(\bar{\omega}_B \times \bar{\beta}_B\right)\right]&\end{aligned} \tag{7}$$

Because the two bodies are connected, Equations 5, 6, and 7 are coupled and must be solved simultaneously. This can be done by rearranging so that the whole set of equations can be treated as a single nine-element vector differential equation, which is then inverted to solve for the state derivatives. To do this, a common coordinate system must be chosen for all three equations. The simplest choice is the body frame. Second, all vector cross products to the left of the equal sign must be replaced by matrix products. In general, a vector cross product can be replaced by a matrix product as follows:

$$\bar{a} \times \bar{b} = \tilde{a}\bar{b}$$

where

$$\tilde{a} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Also in general,

$$\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$$

These two rules must be applied to the left side of the equal sign, but for aesthetics, they are also applied to the right side. The result after rearranging is

$$
\begin{bmatrix} m_V\tilde{I} & -m_C\tilde{\beta}_B^B & -m_C\tilde{\beta}_C^B \\ m_C\tilde{\beta}_B^B & \left(\tilde{I}_B^B - m_C\tilde{\beta}_B^{B\,2}\right) & -m_C\tilde{\beta}_B^B\tilde{\beta}_C^B \\ m_C\tilde{\beta}_C^B & -m_C\tilde{\beta}_C^B\tilde{\beta}_B^B & \left(\tilde{I}_C^B - m_C\tilde{\beta}_C^{B\,2}\right) \end{bmatrix}
\begin{Bmatrix} \dot{\bar{v}}_{B/G}^B\big|_B \\ \dot{\bar{\omega}}_B^B \\ \dot{\bar{\omega}}_C^B \end{Bmatrix}
$$

$$
= \begin{Bmatrix} \bar{F}_B^B + \bar{F}_C^B + m_V\bar{G}^B \\ \bar{M}_B^B + \tilde{\beta}_B^B\left(\bar{F}_C^B + m_C\bar{G}^B\right) \\ \bar{M}_C^B + \tilde{\beta}_C^B\left(\bar{F}_C^B + m_C\bar{G}^B\right) \end{Bmatrix} + \begin{Bmatrix} 0 \\ \bar{M}_h^B \\ -\bar{M}_h^B \end{Bmatrix} - \begin{Bmatrix} 0 \\ \tilde{\omega}_B^B\tilde{I}_B^B\bar{\omega}_B^B \\ \tilde{\omega}_C^B\tilde{I}_C^B\bar{\omega}_C^B \end{Bmatrix}
$$

$$
-\begin{Bmatrix} m_C\left(\tilde{\omega}_C^B\tilde{\omega}_C^B\bar{\beta}_C^B + \tilde{\omega}_B^B\tilde{\omega}_B^B\bar{\beta}_B^B\right) + m_V\left(\tilde{\omega}_B^B\bar{v}_{B/G}^B + 2\tilde{\omega}_G^B\bar{v}_{B/G}^B + \tilde{\omega}_G^B\tilde{\omega}_G^B\bar{x}_B^B\right) \\ m_C\tilde{\beta}_B^B\left(\tilde{\omega}_C^B\tilde{\omega}_C^B\bar{\beta}_C^B + \tilde{\omega}_B^B\tilde{\omega}_B^B\bar{\beta}_B^B + \tilde{\omega}_{B/G}^B\bar{v}_{B/G}^B + 2\tilde{\omega}_G^B\bar{v}_{B/G}^B + \tilde{\omega}_G^B\tilde{\omega}_G^B\bar{x}_B^B\right) \\ m_C\tilde{\beta}_C^B\left(\tilde{\omega}_C^B\tilde{\omega}_C^B\bar{\beta}_C^B + \tilde{\omega}_B^B\tilde{\omega}_B^B\bar{\beta}_B^B + \tilde{\omega}_{B/G}^B\bar{v}_{B/G}^B + 2\tilde{\omega}_G^B\bar{v}_{B/G}^B + \tilde{\omega}_G^B\tilde{\omega}_G^B\bar{x}_B^B\right) \end{Bmatrix} \quad (8)
$$

The matrix premultiplying the derivative vector serves as an expanded inertia tensor. Written in partitioned form above, it appears to be nearly skew symmetric. However, when (tediously) expanded, it is symmetric.

To make use of this equation, each vector in Equation 8 must be written explicitly in the body frame. Selected vectors are listed below. Figures 1 and 2 and the nomenclature section provide further information.

Body CG location with respect to the hinge and hinge location with respect to the combustor CG:

$$
\bar{\beta}_B^B = \begin{Bmatrix} -(l_B - x_B) \\ 0 \\ 0 \end{Bmatrix}, \quad \bar{\beta}_C^B = [T_{C2B}]\begin{Bmatrix} -x_C \\ 0 \\ 0 \end{Bmatrix}
$$

Gravitational acceleration:

$$\overline{G}^B = [T_{H2B}] g_0 \left(\frac{R_E}{R_E + h} \right)^2 \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

Body position: The position of the body CG, $\bar{x}_B$, is relative to the inertial reference point at the Earth's center. Thus, the z-component of $\bar{x}_B{}^H$ (body CG position expressed in the local horizontal frame) is $(R_E + h)$.

Angular velocity of the ground frame:

$$\overline{\omega}_G^B = [T_{G2B}] \begin{Bmatrix} \omega_E \\ 0 \\ 0 \end{Bmatrix} = [T_{H2B}][T_{G2H}] \begin{Bmatrix} \omega_E \\ 0 \\ 0 \end{Bmatrix}$$

Angular velocity of local horizontal frame with respect to the ground frame:

$$\omega_{H/G}^B = [T_{H2B}] \left[\begin{Bmatrix} 0 \\ -\dot{\xi} \\ 0 \end{Bmatrix} + [\xi]_y \begin{Bmatrix} \dot{\lambda} \\ 0 \\ 0 \end{Bmatrix} \right]$$

Combustor inertia tensor: $\tilde{I}_C^B = [T_{B2C}] \tilde{I}_C^C$

KINEMATICS

In the preceding section, nine of the 18 differential equations describing the motion of the two connected bodies were developed. Additional differential equations that describe the position and attitude of the vehicle are now needed. Start with the attitude of the main body with respect to local horizontal. Remember, on the spherical Earth the local horizontal frame rotates with respect to the inertial frame. The body frame is related to

local horizontal by the Euler rotation sequence ψ about z, θ about y, and ϕ about x. The inertial angular velocity of the body is

$$\overline{\omega}_B = \overline{\omega}_{B/H} + \overline{\omega}_{H/G} + \overline{\omega}_G$$

$$\overline{\omega}_{B/H} = \overline{\omega}_B - \overline{\omega}_{H/G} - \overline{\omega}_G$$

Again, use the body frame.

$$\overline{\omega}^B_{B/H} = \begin{Bmatrix} \dot{\phi} \\ 0 \\ 0 \end{Bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{Bmatrix} 0 \\ \dot{\theta} \\ 0 \end{Bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\psi} \end{Bmatrix}$$

$$\overline{\omega}^B_{B/H} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix}$$

Solve for the Euler angle rates using Cramer's rule:

$$\begin{Bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{Bmatrix} = \begin{bmatrix} 1 & \tan\theta\sin\phi & \tan\theta\cos\phi \\ 0 & \cos\phi & -\sin\phi \\ 0 & \dfrac{\sin\phi}{\cos\theta} & \dfrac{\cos\phi}{\cos\theta} \end{bmatrix} \overline{\omega}^B_{B/H} \tag{9}$$

An important drawback exists to using the differential equations for Euler rotation sequence to describe the body angular motion. There is a singularity in the transformation matrix at $\theta = 90$ deg, i.e., when the body is vertical. An alternate way to relate the inertial and the horizontal frames is by using quaternions. This method yields a transformation matrix that has no singularities and that can be calculated more rapidly by

computer. However, quaternions are not physically intuitive, and this vehicle should never reach a large pitch angle, so Euler angles were used in this development.

Next, determine the position on the surface of the spherical Earth of the body CG with respect to a reference point, usually the intersection of the prime meridian and the equator.

$$\bar{x}_{B/G} = \bar{x}_{H/G} + \bar{x}_{B/H}$$
$$\bar{x}_{B/G} = (\bar{x}_H - \bar{x}_G) + \bar{x}_{B/H}$$

Note that the origins of the body and local horizontal frames coincide, so $\bar{x}_{B/H} = 0$. Differentiate with respect to the ground frame:

$$\bar{v}_{B/G} = \dot{\bar{x}}_{B/G}\big|_G$$
$$\bar{v}_{B/G} = \dot{\bar{x}}_H\big|_G - \dot{\bar{x}}_G\big|_G$$

Assume the Earth rotates at a constant rate about the axis through the poles, $\dot{\bar{x}}_G\big|_G = 0$.

$$\bar{v}_{B/G} = \dot{\bar{x}}_H\big|_H + \bar{\omega}_{H/G} \times \bar{x}_H$$

To solve for the latitude and longitude rates, this equation must be expressed in scalar form. Choose the local horizontal frame:

$$\bar{v}^H_{B/G} = \dot{\bar{x}}^H_B\big|_H + \bar{\omega}^H_{H/G} \times \bar{x}^H_H$$
$$\bar{v}^H_{B/G} = [T_{B2H}]\bar{v}^B_{B/G}$$

Note that for a spherical earth, $\bar{v}^H_{B/G}$ is a non-inertial quantity. In a later section, when the equations are reduced to flat Earth, it will be inertial. To avoid confusion with the

inertial velocity, we will add the superscript "H" to each element of the vector, keeping in mind that the two become identical for flat Earth.

$$\bar{v}_{B/G}^{H} = \begin{Bmatrix} u^H \\ v^H \\ w^H \end{Bmatrix}$$

$$\begin{Bmatrix} u^H \\ v^H \\ w^H \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -\dot{h} \end{Bmatrix} + \begin{Bmatrix} \dot{\lambda}\cos\xi \\ -\dot{\xi} \\ -\dot{\lambda}\sin\xi \end{Bmatrix} \times \begin{Bmatrix} 0 \\ 0 \\ -(R_E + h) \end{Bmatrix}$$

Solving for the spherical coordinate rates:

$$\dot{\xi} = \frac{u^H}{R_E + h} \tag{10}$$

$$\dot{\lambda} = \frac{v^H}{(R_E + h)\cos\xi} \tag{11}$$

$$\dot{h} = -w^H \tag{12}$$

The geographic position is

$$X = R_E \xi$$

$$Y = R_E \lambda$$

The final equations needed are for the combustor deflection relative to the body.

$$\bar{\omega}_{C/B} = \bar{\omega}_C - \bar{\omega}_B$$

Write out in the body frame.

$$\begin{Bmatrix} \dot{\delta}_x \cos\delta_y \cos\delta_z - \dot{\delta}_y \sin\delta_z \\ \dot{\delta}_x \cos\delta_y \sin\delta_z + \dot{\delta}_y \cos\delta_z \\ -\dot{\delta}_x \sin\delta_y + \dot{\delta}_z \end{Bmatrix} = \overline{\omega}_C^B - \overline{\omega}_B^B = \begin{Bmatrix} p \\ q \\ r \end{Bmatrix}_{C/B} \tag{13}$$

By convention, p, q, and r are assumed to be angular rates about the axes of the body frame. $\overline{\omega}_B^B$ and $\overline{\omega}_C^B$ are available from integrating Equations 8. Solving for the combustor deflections yields

$$\begin{Bmatrix} \dot{\delta}_x \\ \dot{\delta}_y \\ \dot{\delta}_z \end{Bmatrix} = \begin{bmatrix} 1 & -\cos\delta_y \tan\delta_z & \sin\delta_y \\ \tan\delta_z & \cos\delta_y & \sin\delta_y \tan\delta_z \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} p \\ q \\ r \end{Bmatrix}_{C/B} \tag{14}$$

Remember, the combustor deflection angles are modeled as an Euler sequence in this development. This choice was made for convenience in the formulation and would represent a set of nested gimbals. Depending on how the actual servo is designed, some other model might be more appropriate.

The combustor CG velocity is available algebraically. Though it is not needed for simulation of the vehicle's motion, it is provided here for reference. Remember, the subscript "h" refers to the hinge.

$$\begin{aligned} \overline{v}_{C/G} &= \overline{v}_{C/B} + \overline{v}_{B/G} \\ &= \overline{v}_{C/h} + \overline{v}_{h/B} + \overline{v}_{B/G} \\ &= \overline{\omega}_{C/G} \times \overline{\beta}_C + \overline{\omega}_{B/G} \times \overline{\beta}_B + \overline{v}_{B/G} \end{aligned}$$

For interest, the surface distance flown along a great circle since launch is also included (Reference 2):

$$R = R_E \cos^{-1}\left\{\cos\left(\frac{\pi}{2}-\xi_2\right)\cos\left(\frac{\pi}{2}-\xi_1\right)+\sin\left(\frac{\pi}{2}-\xi_2\right)\sin\left(\frac{\pi}{2}-\xi_1\right)\cos(\lambda_2-\lambda_1)\right\}$$

The motion of the missile and combustor have been fully described. Equations 8 through 12 and Equation 14 describe the dynamics of the vehicle's 18 independent states, subject to the following assumptions:

1. Combustor and body are rigid
2. Control servo in the hinge only exerts pure couple
3. Hinge can be modeled as a gimbal set
4. Earth is perfectly spherical, rotating at a constant rate
5. Mass properties of the two vehicle sections change "slowly"

GENERAL SIX-DEGREE-OF-FREEDOM MOTION OVER SPHERICAL EARTH WITH NO-ROLL HINGE

Adding the constraint that the combustor may not roll with respect to the body removes a degree of freedom; the combustor roll rate, $\omega_{Cx}^{B} = p_C$, is no longer independent and is deleted from the state vector. Further, M_{hx}^{B} is just a reaction moment and is eliminated from the equations. Doing this will reduce the size of the state vector to be integrated, but will also destroy the nice, compact form of Equation 8.

Define p_C using Equation 13, remembering that $\delta_x = \dot{\delta}_x = 0$.

$$\begin{Bmatrix} -\dot{\delta}_y \sin\delta_z \\ \dot{\delta}_y \cos\delta_z \\ \dot{\delta}_z \end{Bmatrix} = \begin{Bmatrix} p \\ q \\ r \end{Bmatrix}_{C/B}$$

The y and z elements yield differential equations for δ_y and δ_z.

$$\dot{\delta}_y = \frac{q_{C/B}}{\cos\delta_z} \tag{15}$$

$$\dot{\delta}_z = r_{C/B}. \tag{16}$$

Use the x element to relate p_C to the other states.

$$p_{C/B} = p_C - p_B = -\dot{\delta}_y \sin\delta_z$$

$$p_C = p_B - \dot{\delta}_y \sin\delta_z$$

$$p_C = p_B - (q_C - q_B)\tan\delta_z \tag{17}$$

Differentiate and use the expression for $\dot{\delta}_z$.

$$\dot{p}_C = \dot{p}_B - (\dot{q}_C - \dot{q}_B)\tan\delta_z - \frac{(q_C - q_B)(r_C - r_B)}{\cos^2\delta_z} \tag{18}$$

Set up the following definitions to make subsequent work more compact.

$$\begin{Bmatrix} \bar{k}_v \\ \bar{k}_b \\ \bar{k}_c \end{Bmatrix} = -\begin{Bmatrix} 0 \\ \tilde{\omega}_B^B \tilde{I}_B^B \overline{\omega}_B^B \\ \tilde{\omega}_C^B \tilde{I}_C^B \overline{\omega}_C^B \end{Bmatrix} - \begin{Bmatrix} m_C\left(\tilde{\omega}_C^B \tilde{\omega}_C^B \overline{\beta}_C^B + \tilde{\omega}_B^B \tilde{\omega}_B^B \overline{\beta}_B^B\right) + m_V\left(\tilde{\omega}_B^B \bar{v}_{B/G}^B + 2\tilde{\omega}_G^B \bar{v}_{B/G}^B + \tilde{\omega}_G^B \tilde{\omega}_G^B \bar{x}_B^B\right) \\ m_C \tilde{\beta}_B^B\left(\tilde{\omega}_C^B \tilde{\omega}_C^B \overline{\beta}_C^B + \tilde{\omega}_B^B \tilde{\omega}_B^B \overline{\beta}_B^B + \tilde{\omega}_{B/G}^B \bar{v}_{B/G}^B + 2\tilde{\omega}_G^B \bar{v}_{B/G}^B + \tilde{\omega}_G^B \tilde{\omega}_G^B \bar{x}_B^B\right) \\ m_C \tilde{\beta}_C^B\left(\tilde{\omega}_C^B \tilde{\omega}_C^B \overline{\beta}_C^B + \tilde{\omega}_B^B \tilde{\omega}_B^B \overline{\beta}_B^B + \tilde{\omega}_{B/G}^B \bar{v}_{B/G}^B + 2\tilde{\omega}_G^B \bar{v}_{B/G}^B + \tilde{\omega}_G^B \tilde{\omega}_G^B \bar{x}_B^B\right) \end{Bmatrix}$$

(19a)

$$\begin{Bmatrix} \bar{f}_v \\ \bar{f}_b \\ \bar{f}_c \end{Bmatrix} = \begin{Bmatrix} \bar{F}_B^B + \bar{F}_C^B + m_V \bar{G}^B \\ \bar{M}_B^B + \tilde{\beta}_B^B\left(\bar{F}_C^B + m_C \bar{G}^B\right) \\ \bar{M}_C^B + \tilde{\beta}_C^B\left(\bar{F}_C^B + m_C \bar{G}^B\right) \end{Bmatrix} \tag{19b}$$

$$\begin{bmatrix} \tilde{J}11 & \tilde{J}12 & \tilde{J}13 \\ \tilde{J}21 & \tilde{J}22 & \tilde{J}23 \\ \tilde{J}31 & \tilde{J}32 & \tilde{J}33 \end{bmatrix} = \begin{bmatrix} m_V \tilde{I} & -m_C \tilde{\beta}_B^B & -m_C \tilde{\beta}_C^B \\ m_C \tilde{\beta}_B^B & \left(\bar{I}_B^B - m_C \tilde{\beta}_B^{B\,2}\right) & -m_C \tilde{\beta}_B^B \tilde{\beta}_C^B \\ m_C \tilde{\beta}_C^B & -m_C \tilde{\beta}_C^B \tilde{\beta}_B^B & \left(\bar{I}_C^B - m_C \tilde{\beta}_C^{B\,2}\right) \end{bmatrix} \tag{19c}$$

Equation 8 then becomes

$$\begin{bmatrix} \tilde{J}11 & \tilde{J}12 & \tilde{J}13 \\ \tilde{J}21 & \tilde{J}22 & \tilde{J}23 \\ \tilde{J}31 & \tilde{J}32 & \tilde{J}33 \end{bmatrix} \begin{Bmatrix} \dot{\bar{v}}_{B/G}^B\big|_B \\ \dot{\bar{\omega}}_B^B \\ \dot{\bar{\omega}}_C^B \end{Bmatrix} = \begin{Bmatrix} \bar{f}_v \\ \bar{f}_b \\ \bar{f}_c \end{Bmatrix} + \begin{Bmatrix} 0 \\ \bar{M}_h^B \\ -\bar{M}_h^B \end{Bmatrix} + \begin{Bmatrix} \bar{k}_v \\ \bar{k}_b \\ \bar{k}_c \end{Bmatrix} \tag{20}$$

Remember,

$$\begin{Bmatrix} \dot{\bar{v}}_{B/G}^B\big|_B \\ \dot{\bar{\omega}}_B^B \\ \dot{\bar{\omega}}_C^B \end{Bmatrix} = \left\{ \dot{u}_{B/G}^B\big|_B \quad \dot{v}_{B/G}^B\big|_B \quad \dot{w}_{B/G}^B\big|_B \;\vdots\; \dot{p}_B \quad \dot{q}_B \quad \dot{r}_B \;\vdots\; \dot{p}_C \quad \dot{q}_C \quad \dot{r}_C \right\}^T$$

$$\begin{Bmatrix} 0 \\ \bar{M}_h^B \\ -\bar{M}_h^B \end{Bmatrix} = \left\{ 0 \quad 0 \quad 0 \;\vdots\; M_{hx}^B \quad M_{hy}^B \quad M_{hz}^B \;\vdots\; -M_{hx}^B \quad -M_{hy}^B \quad -M_{hz}^B \right\}^T$$

Solve the seventh equation in Equation 20 for M_{hx}^B.

$$\left[J31_{11} \dot{u}_{B/G}^B\big|_B + J31_{12} \dot{v}_{B/G}^B\big|_B + J31_{13} \dot{w}_{B/G}^B\big|_B\right] + \left[J32_{11} \dot{p}_B + J32_{12} \dot{q}_B + J32_{13} \dot{r}_B\right]$$
$$+\left[J33_{11} \dot{p}_C + J33_{12} \dot{q}_C + J33_{13} \dot{r}_C\right] = f_{Cx} - M_{hx}^B + k_{Cx}$$

$$M_{hx}^B = -\left[J31_{11}\dot{u}_{B/G}^B\Big|_B + J31_{12}\dot{v}_{B/G}^B\Big|_B + J31_{13}\dot{w}_{B/G}^B\Big|_B\right] - \left[J32_{11}\dot{p}_B + J32_{12}\dot{q}_B + J32_{13}\dot{r}_B\right]$$
$$-\left[J33_{11}\dot{p}_C + J33_{12}\dot{q}_C + J33_{13}\dot{r}_C\right] + f_{Cx} + k_{Cx}$$

Substitute into the fourth differential equation in Equation 20

$$\left[J21_{11}\dot{u}_{B/G}^B\Big|_B + J21_{12}\dot{v}_{B/G}^B\Big|_B + J21_{13}\dot{w}_{B/G}^B\Big|_B\right] + \left[J22_{11}\dot{p}_B + J22_{12}\dot{q}_B + J22_{13}\dot{r}_B\right]$$
$$+\left[J23_{11}\dot{p}_C + J23_{12}\dot{q}_C + J23_{13}\dot{r}_C\right] = f_{Bx} + M_{hx}^B + k_{Bx}$$

$$\left[J21_{11}\dot{u}_{B/G}^B\Big|_B + J21_{12}\dot{v}_{B/G}^B\Big|_B + J21_{13}\dot{w}_{B/G}^B\Big|_B\right] + \left[J22_{11}\dot{p}_B + J22_{12}\dot{q}_B + J22_{13}\dot{r}_B\right]$$
$$+\left[J23_{11}\dot{p}_C + J23_{12}\dot{q}_C + J23_{13}\dot{r}_C\right]$$
$$= -\left[J31_{11}\dot{u}_{B/G}^B\Big|_B + J31_{12}\dot{v}_{B/G}^B\Big|_B + J31_{13}\dot{w}_{B/G}^B\Big|_B\right]$$
$$-\left[J32_{11}\dot{p}_B + J32_{12}\dot{q}_B + J32_{13}\dot{r}_B\right] - \left[J33_{11}\dot{p}_C + J33_{12}\dot{q}_C + J33_{13}\dot{r}_C\right]$$
$$+f_{Bx} + f_{Cx} + k_{Bx} + k_{Cx}$$

Place all the derivatives on the left side and write out the full matrix equation. The result is shown in Equation 22 at the end of this section. To eliminate the extra state p_C, insert Equation 18 into Equation 22 and rearrange. The result is shown in Equation 23 at the end of this section.

Now, for the specific vehicle being modeled, this equation can be simplified. First, the deflection angles are fairly small, and second, the deflection rates are large in only one direction at a time, i.e., $\dot{\delta}_y$ and $\dot{\delta}_z$ will not be large at the same time. Referring to Equation 18,

$$\dot{p}_C = \dot{p}_B - (\dot{q}_C - \dot{q}_B)\tan\delta_z - \frac{(q_C - q_B)(r_C - r_B)}{\cos^2\delta_z}$$

Recall that $\dot{\delta}_y = (q_C - q_B)/\cos\delta_z$ and $\dot{\delta}_z = (r_C - r_B)$.

$$\dot{p}_C = \dot{p}_B - (\dot{q}_C - \dot{q}_B)\sin\delta_z - \frac{\dot{\delta}_y\dot{\delta}_z}{\cos\delta_z}$$

Apply small angle assumption:

$$\dot{p}_C \approx \dot{p}_B - (\dot{q}_C - \dot{q}_B)\delta_z - \dot{\delta}_y\dot{\delta}_z$$

Remember that $\dot{\delta}_y$ and $\dot{\delta}_z$ will not be large at the same time:

$$\dot{p}_C \approx \dot{p}_B \tag{21}$$

Applying this yields Equation 24.

Equation 22

$$\left[\begin{array}{c} \\ \\ \left[\begin{array}{ccc:ccc:ccc} & \tilde{J}11 & & & \tilde{J}12 & & & \tilde{J}13 & \\ \hdashline & \tilde{J}21 & & & \tilde{J}22 & & & \tilde{J}23 & \\ \hdashline J31_{21} & J31_{22} & J31_{23} & J32_{21} & J32_{22} & J32_{23} & J33_{21} & J33_{22} & J33_{23} \\ J31_{31} & J31_{32} & J31_{33} & J32_{31} & J32_{32} & J32_{33} & J33_{31} & J33_{32} & J33_{33} \end{array}\right] \\ \\ \\ \end{array}\right. + \left[\begin{array}{ccc:c} 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ \hdashline J31_{11} & J31_{12} & J31_{13} & J \\ 0 & 0 & 0 & \\ 0 & 0 & 0 & \\ \hdashline 0 & 0 & 0 & \\ 0 & 0 & 0 & \end{array}\right.$$

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$$
\left[\left[\begin{array}{ccc:ccc:ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
J31_{11} & J31_{12} & J31_{13} & J32_{11} & J32_{12} & J32_{13} & J33_{11} & J33_{12} & J33_{13} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]\right]
\left\{\begin{array}{c}
\left.\dot{\bar{v}}^{B}_{B/G}\right|_{B} \\
\dot{\bar{\omega}}^{B}_{B} \\
\dot{\bar{\omega}}^{B}_{C}
\end{array}\right\}
=
\left\{\begin{array}{c}
\bar{f}_V \\
\hdashline
\bar{\bar{f}}_B \\
\hdashline
f_{C\,y} \\
f_{C\,z}
\end{array}\right\}
+
\left\{\begin{array}{c}
0 \\
\hdashline
f_{C\,x} + k_{C\,x} \\
M^{B}_{h\,y} \\
M^{B}_{h\,z} \\
\hdashline
-M^{B}_{h\,y} \\
-M^{B}_{h\,z}
\end{array}\right\}
+
\left\{\begin{array}{c}
\bar{k}_V \\
\hdashline
\bar{\bar{k}}_B \\
\hdashline
k_{C\,y} \\
k_{C\,z}
\end{array}\right\}
$$

Equation 23

$$
\left[\left[\begin{array}{ccc:ccc:cc}
 & & & & & & J13_{12} & J13_{13} \\
 & \bar{J}11 & & & \bar{J}12 & & J13_{22} & J13_{23} \\
 & & & & & & J13_{32} & J13_{33} \\
\hdashline
 & & & & & & J23_{12} & J23_{13} \\
 & \bar{J}21 & & & \bar{J}22 & & J23_{22} & J23_{23} \\
 & & & & & & J23_{32} & J23_{33} \\
\hdashline
J31_{21} & J31_{22} & J31_{23} & J32_{21} & J32_{22} & J32_{23} & J33_{22} & J33_{23} \\
J31_{31} & J31_{32} & J31_{33} & J32_{31} & J32_{32} & J32_{33} & J33_{32} & J33_{33}
\end{array}\right] + \left[\begin{array}{ccc:c}
0 & 0 & 0 & J13_{11} \\
0 & 0 & 0 & J13_{21} \\
0 & 0 & 0 & J13_{31} \\
\hdashline
J31_{11} & J31_{12} & J31_{13} & \left[\begin{array}{c} J32_{11} \\ +\left(J23_{11}+J33_{11}\right) \end{array}\right] \\
0 & 0 & 0 & J23_{21} \\
0 & 0 & 0 & J23_{31} \\
\hdashline
0 & 0 & 0 & J33_{21} \\
0 & 0 & 0 & J33_{31}
\end{array}\right]\right.
$$

$$
-\left\{\begin{array}{c} \bar{f}_V \\ \hdashline \bar{f}_B \\ \hdashline f_{C\,y} \\ f_{C\,z} \end{array}\right\}
+\left\{\begin{array}{c} 0 \\ \hdashline f_{C\,x}+k_{C\,x} \\ M_{h\,y}^{B} \\ M_{h\,z}^{B} \\ \hdashline -M_{h\,y}^{B} \\ -M_{h\,z}^{B} \end{array}\right\}
+\left\{\begin{array}{c} \bar{k}_V \\ \hdashline \bar{k}_B \\ \hdashline k_{C\,y} \\ k_{C\,z} \end{array}\right\}
+\frac{\left(q_C-q_B\right)\left(r_C-r_B\right)}{\cos^2\delta_z}\left\{\begin{array}{c} J13_{11} \\ J13_{21} \\ J13_{31} \\ \hdashline J23_{11}+J33_{11} \\ J23_{21} \\ J23_{31} \\ \hdashline J33_{21} \\ J33_{31} \end{array}\right\}
$$

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$$
\left.\begin{array}{ccc:cc}
J13_{11} & J13_{11}\tan\delta_z & 0 & -J13_{11}\tan\delta_z & 0 \\
J13_{21} & J13_{21}\tan\delta_z & 0 & -J13_{21}\tan\delta_z & 0 \\
J13_{31} & J13_{31}\tan\delta_z & 0 & -J13_{31}\tan\delta_z & 0 \\
\hdashline
\begin{bmatrix} J32_{11} \\ +\left(J23_{11}+J33_{11}\right) \end{bmatrix} & \begin{bmatrix} J32_{12} \\ +\left(J23_{11}+J33_{11}\right)\tan\delta_z \end{bmatrix} & J32_{13} & \begin{bmatrix} J33_{12} \\ -\left(J23_{11}+J33_{11}\right)\tan\delta_z \end{bmatrix} & J33_{13} \\
J23_{21} & J23_{21}\tan\delta_z & 0 & -J23_{21}\tan\delta_z & 0 \\
J23_{31} & J23_{31}\tan\delta_z & 0 & -J23_{31}\tan\delta_z & 0 \\
\hdashline
J33_{21} & J33_{21}\tan\delta_z & 0 & -J33_{21}\tan\delta_z & 0 \\
J33_{31} & J33_{31}\tan\delta_z & 0 & -J33_{31}\tan\delta_z & 0
\end{array}\right]\,\Bigg]
\begin{Bmatrix} \left.\dot{\vec{v}}^{B}_{B/G}\right|_B \\ \hdashline \dot{p}_B \\ \dot{q}_B \\ \dot{r}_B \\ \hdashline \dot{q}_C \\ \dot{r}_C \end{Bmatrix}
$$

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Equation 24

$$
\left[\left[\begin{array}{ccc:ccc:cc}
 & & & & & & J13_{12} & J13_{13} \\
 & \bar{J}11 & & & \bar{J}12 & & J13_{22} & J13_{23} \\
 & & & & & & J13_{32} & J13_{33} \\
\hdashline
 & & & & & & J23_{12} & J23_{13} \\
 & \bar{J}21 & & & \bar{J}22 & & J23_{22} & J23_{23} \\
 & & & & & & J23_{32} & J23_{33} \\
\hdashline
J31_{21} & J31_{22} & J31_{23} & J32_{21} & J32_{22} & J32_{23} & J33_{22} & J33_{23} \\
J31_{31} & J31_{32} & J31_{33} & J32_{31} & J32_{32} & J32_{33} & J33_{32} & J33_{33}
\end{array}\right]
+
\left[\begin{array}{ccc:c}
0 & 0 & 0 & J13_{11} \\
0 & 0 & 0 & J13_{21} \\
0 & 0 & 0 & J13_{31} \\
\hdashline
J31_{11} & J31_{12} & J31_{13} & J32_{11} + J23_{11} + J33_{11} \\
0 & 0 & 0 & J23_{21} \\
0 & 0 & 0 & J23_{31} \\
\hdashline
0 & 0 & 0 & J33_{21} \\
0 & 0 & 0 & J33_{31}
\end{array}\right]\right.
$$

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$$
\left.\begin{array}{ccc:cc}
J13_{11} & 0 & 0 & 0 & 0 \\
J13_{21} & 0 & 0 & 0 & 0 \\
J13_{31} & 0 & 0 & 0 & 0 \\
\hdashline
{}_{11} + J23_{11} + J33_{11} & J32_{12} & J32_{13} & J33_{12} & J33_{13} \\
J23_{21} & 0 & 0 & 0 & 0 \\
J23_{31} & 0 & 0 & 0 & 0 \\
\hdashline
J33_{21} & 0 & 0 & 0 & 0 \\
J33_{31} & 0 & 0 & 0 & 0
\end{array}\right]\Bigg]
\begin{Bmatrix} \dot{\bar{v}}^{B}_{B/G}\big|_B \\ \hdashline \dot{p}_B \\ \dot{q}_B \\ \dot{r}_B \\ \hdashline \dot{q}_C \\ \dot{r}_C \end{Bmatrix}
=
\begin{Bmatrix} \bar{f}_V \\ \hdashline \bar{\bar{f}}_B \\ \hdashline f_{C\,y} \\ f_{C\,z} \end{Bmatrix}
+
\begin{Bmatrix} 0 \\ \hdashline f_{C\,x} + k_{C\,x} \\ M^{B}_{h\,y} \\ M^{B}_{h\,z} \\ \hdashline -M^{B}_{h\,y} \\ -M^{B}_{h\,z} \end{Bmatrix}
+
\begin{Bmatrix} \bar{k}_V \\ \hdashline \bar{\bar{k}}_B \\ \hdashline k_{C\,y} \\ k_{C\,z} \end{Bmatrix}
$$

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SPECIAL CASE: SINGLE RIGID BODY

When the second body is removed, Equations 8 decouple and can be treated as three separate vector equations, rather than as one 9-element equation. Further, the differential equations for $\tilde{\omega}_C$, δ_x, δ_y, and δ_z are typically replaced by three second-order equations to adequate fidelity. This is the case for most missiles, both thrust vector controlled and aerodynamically controlled, because the mass of the deflecting nozzle or control surface is negligible relative to the total mass of the rest of the vehicle, and the distance from the hinge to the CG is small. The effect of the second body on the dynamics is restricted to its contribution to the forces and moments on the body. From Equation 8, the equations for the body velocity and angular rate are

$$m_V \dot{\bar{v}}^B_{B/G}\Big|_B = \bar{F}^B_B + \bar{F}^B_C + m_V \bar{G}^B + m_V\left(\tilde{\omega}^B_B \bar{v}^B_{B/G} + 2\tilde{\omega}^B_G \bar{v}^B_{B/G} + \tilde{\omega}^B_G \tilde{\omega}^B_G \bar{x}^B_B\right)$$

$$\tilde{I}^B_B \dot{\bar{\omega}}^B_B = \bar{M}^B_B + \tilde{\beta}^B_B \bar{F}^B_C + \bar{M}^B_h - \tilde{\omega}^B_B \tilde{I}^B_B \bar{\omega}^B_B$$

The remaining Equations 9 though 12 are unchanged.

SPECIAL CASE: FLAT EARTH

For flat Earth, the radius of the Earth becomes infinite and its angular rate zero. The gravitational field is usually assumed to be uniform, and the local horizontal frame becomes parallel to the ground frame, which is now inertial. Assuming the combustor can roll with respect to the main body, most of these changes can be applied to the initial set of equations by zeroing out the appropriate terms.

$$\begin{bmatrix} m_V \bar{I} & -m_C \tilde{\beta}_B^B & -m_C \tilde{\beta}_C^B \\ m_C \tilde{\beta}_B^B & \left(\bar{I}_B^B - m_C \tilde{\beta}_B^{B\,2}\right) & -m_C \tilde{\beta}_B^B \tilde{\beta}_C^B \\ m_C \tilde{\beta}_C^B & -m_C \tilde{\beta}_C^B \tilde{\beta}_B^B & \left(\bar{I}_C^B - m_C \tilde{\beta}_C^{B\,2}\right) \end{bmatrix} \begin{Bmatrix} \dot{\bar{v}}_{B/G}^B\big|_B \\ \dot{\bar{\omega}}_B^B \\ \dot{\bar{\omega}}_C^B \end{Bmatrix} = \begin{Bmatrix} \overline{F}_B^B + \overline{F}_C^B + m_V \overline{G}^B \\ \overline{M}_B^B + \tilde{\beta}_B^B\left(\overline{F}_C^B + m_C \overline{G}^B\right) \\ \overline{M}_C^B + \tilde{\beta}_C^B\left(\overline{F}_C^B + m_C \overline{G}^B\right) \end{Bmatrix}$$

$$+\begin{Bmatrix} 0 \\ \overline{M}_h^B \\ -\overline{M}_h^B \end{Bmatrix} - \begin{Bmatrix} 0 \\ \tilde{\omega}_B^B \bar{I}_B^B \overline{\omega}_B^B \\ \tilde{\omega}_C^B \bar{I}_C^B \overline{\omega}_C^B \end{Bmatrix} - \begin{Bmatrix} m_C\left(\tilde{\omega}_C^B \tilde{\omega}_C^B \overline{\beta}_C^B + \tilde{\omega}_B^B \tilde{\omega}_B^B \overline{\beta}_B^B\right) + m_V \tilde{\omega}_B^B \overline{v}_{B/G}^B \\ m_C \tilde{\beta}_B^B\left(\tilde{\omega}_C^B \tilde{\omega}_C^B \overline{\beta}_C^B + \tilde{\omega}_B^B \tilde{\omega}_B^B \overline{\beta}_B^B + \tilde{\omega}_{B/G}^B \overline{v}_{B/G}^B\right) \\ m_C \tilde{\beta}_C^B\left(\tilde{\omega}_C^B \tilde{\omega}_C^B \overline{\beta}_C^B + \tilde{\omega}_B^B \tilde{\omega}_B^B \overline{\beta}_B^B + \tilde{\omega}_{B/G}^B \overline{v}_{B/G}^B\right) \end{Bmatrix} \qquad (25)$$

The exception is the set of equations describing the missile CG position. In this case it is necessary to shift from motion on the surface of a sphere to motion over a flat surface. The missile position rate in the inertial frame is simply the velocity in the body frame, transformed, or

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{Bmatrix}_B = \left[T_{B2H}\right] \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}_B \qquad (26)$$

where $\dot{h} = -w_B$

The surface distance traveled is simply

$$R = \sqrt{x_B^2 + y_B^2} \tag{27}$$

Equations 9 and 14 are unaffected.

In this section, the combustor can roll with respect to the body. For the LDR, however, this roll freedom is gone, so equations 9, 15, 16, 19, 24 (with $\bar{\omega}_G$=0) and 26, along with assumption 21, were implemented in the simulation.

SPECIAL CASE: FLAT EARTH AND PITCH PLANE

The assumption that motion is restricted to the pitch plane results in an enormous simplification: the three vector equations in Equation 8 are reduced to three scalar equations, which can be solved separately. The notation for the scalar forces and moments is illustrated in Figure 3.

An additional assumption here is that both sections are symmetric homogeneous cylinders, so the inertia tensors of the body and combustor are diagonal with the y and z elements equal, i.e.,

$$\bar{I}_B^B = \begin{bmatrix} I_{Bx} & 0 & 0 \\ 0 & I_B & 0 \\ 0 & 0 & I_B \end{bmatrix}, \text{ and } \bar{I}_C^C = \begin{bmatrix} I_{Cx} & 0 & 0 \\ 0 & I_C & 0 \\ 0 & 0 & I_C \end{bmatrix}$$

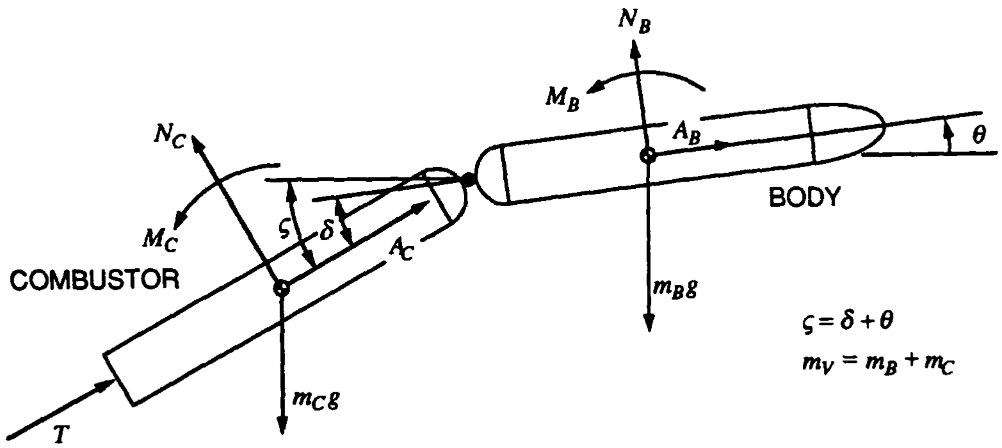

FIGURE 3. Planar Vehicle Over Flat Earth.

Since this is the planar case, neither the body nor the combustor roll.

$$m_V \dot{u} + m_C \ddot{\zeta} x_C \sin\delta = A_B + (A_C + T)\cos\delta - N_C \sin\delta$$
$$-m_V g_0 \sin\theta - m_C\left[\dot{\theta}^2 (l - x_B) + \dot{\zeta}^2 x_C \cos\delta\right] - m_V \dot{\theta} w$$

$$m_V \dot{w} + m_C \ddot{\theta}(l - x_B) + m_C \ddot{\zeta} x_C \cos\delta = -N_B - N_C \cos\delta - (A_C + T)\sin\delta$$
$$+m_V g_0 \cos\theta + m_C \dot{\zeta}^2 x_C \sin\delta + m_V \dot{\theta} u$$

$$I_B \ddot{\theta} + m_C (l - x_B)\left[\dot{w} + \ddot{\theta}(l - x_B) + \ddot{\zeta} x_C \cos\delta\right] = M_B + M_h$$
$$+(l - x_B)\left[-N_C \cos\delta - (A_C + T)\sin\delta + m_C g_0 \cos\theta + m_C\left(\dot{\zeta}^2 x_C \sin\delta - \dot{\theta} u\right)\right]$$

$$I_C \ddot{\zeta} + m_C x_C \left[\dot{w}\cos\delta + \dot{u}\sin\delta + \ddot{\theta}(l - x_B)\cos\delta + \ddot{\zeta} x_C \right] = M_C - M_h$$
$$+ x_C \left\{ -N_C + m_C g_0 \cos\zeta - m_C \left[\dot{\theta}^2 (l - x_B)\sin\delta + \dot{\theta}(u\cos\delta - w\sin\delta) \right] \right\}$$

Equation 9 becomes $\dot{\theta} = q_B$. The differential equations for position reduce to

$$\dot{x}_B = u_B \cos\theta + w_B \sin\theta$$
$$\dot{z}_B = -u_B \sin\theta + w_B \cos\theta$$

CONCLUSIONS

Equations of motion for a missile consisting of two rigid bodies connected by a hinge have been developed for flight over a spherical rotating Earth. These equations have been specialized to three simpler cases: flight over a spherical rotating Earth where the second body is much smaller than the first, flight over a stationary flat Earth, and pitch plane flight over a flat Earth.

For modeling the dynamics of more complex vehicles, i.e., with more sections or with more complex connections, the development by Abzug in Reference 3 would be useful. The notation, while intricate, is well laid out for handling the complexity of that type of vehicle. For the LDR vehicle, we used both approaches for developing the first set of equations in this report, and concluded that the more intuitive approach presented here was workable for two bodies and would be clearer to present in this report. As the

number of bodies increases and the connections become more complex, the more intuitive approach taken here becomes impractical.

NOMENCLATURE

GENERAL NOTATION

Because this is a two-body vehicle, reference is made to the two portions, the front *body* and the aft *combustor*. These two parts make up the *vehicle*.

[ ] - matrix

$(\tilde{\ })$ - (tilde overbar) - matrix

$(\bar{\ })$ - (overbar) - vector

Transformations and Coordinate systems

Several coordinate systems are used in this development. First, of course, is the inertial frame, which is assumed to be fixed in inertial space. Next, the ground frame is attached to a fixed point on the surface of the Earth. For the flat Earth, the ground and inertial frames are identical. The body frame travels and rotates with the main body. The combustor frame travels and rotates with the combustor. The local horizontal frame travels with the body, but remains aligned with the local horizon. For flat Earth, the local horizontal and ground frames remain aligned.

The coordinate frames are related to each other by a sequence of Euler rotations. In this discussion, all Euler rotations are in the sequence x,y,z. To illustrate with the rotation from local horizontal to body frames:

$$\bar{a}^B = [T_{H2B}]\bar{a}^H$$

where

$$[T_{H2B}] = [\phi]_x[\theta]_y[\psi]_z$$

$$[T_{H2B}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[T_{H2B}] = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix}$$

Other transformations

$[T_{B2C}] = [\delta_x][\delta_y][\delta_z]$ - body to combustor

$[T_{G2H}] = [-\xi]_y[\lambda]_x$ - ground to local horizontal

$[T_{H2B}] = [\phi]_x[\theta]_y[\psi]_z$ - local horizontal to body

Derivatives

A final relevant subject is that of noninertial derivatives. Since there are several noninertial frames in this development, the notation becomes a little more intricate than usual.

$\dfrac{d\bar{a}}{dt}$ - inertial time derivative of vector $\bar{a}$

$\bar{\omega}_{B/G}$ - angular velocity of body frame "B" with respect to the ground frame "G"

$\dot{\bar{a}}\big|_G$ - time derivative of vector $\bar{a}$ with respect to (usually) noninertial frame "G"

Note that $\dot{\bar{a}}_{B/G}\big|_I = \dfrac{d\bar{a}_{B/G}}{dt} = \dot{\bar{a}}_{B/G}\big|_G + \bar{\omega}_G \times \bar{a}_{B/G}$. Since the angular velocity of "G" is with respect to inertial space, the "I" in the subscript usually is dropped, i.e., $\bar{\omega}_{G/I} = \bar{\omega}_G$.

$\bar{a}^B$ - vector $\bar{a}$ expressed in components along the body frame

Note that a vector may be expressed in any coordinate system, regardless of its physical meaning. For example, the angular velocity of the body with respect to the ground ($\bar{\omega}_{B/G}$) may be expressed in components in the body frame ($\bar{\omega}^B_{B/G}$), in components along the ground frame ($\bar{\omega}^G_{B/G}$), or any other frame, such as the combustor frame ($\bar{\omega}^C_{B/G}$). References 1 and 6 both contain good discussions of transformations.

SYMBOLS

Superscripts and subscripts

B - body

C - combustor (motor)

G - ground

h - hinge

H - local horizontal

I - inertial

V - vehicle (body + combustor)

Scalar variables

R_E - radius of Earth

R - distance travelled by main body

h - altitude

X - position from a reference point along constant meridian of the spherical Earth

Y - position from a reference along constant parallel of the spherical Earth

λ - latitude position

ξ - longitude position

m - mass

g_0 - magnitude of gravitational acceleration at sea level

u,v,w - components of velocity in body frame

p,q,r - components of angular velocity in body frame

x,y,z - position components

ω_E - magnitude of Earth's angular velocity

Vector Variables

$\bar{a}$ - a general vector

$\bar{b}$ - a general vector

$\bar{F}$ - external forces except due to gravity and due to the other body via the hinge

$\bar{G}$ - gravitational acceleration

$\bar{M}$ - external moments except due to the other body via the hinge

$\bar{x}$ - position

$\bar{v}$ - velocity

$\bar{a}$ - acceleration

$\bar{\omega}$ - angular velocity

$\bar{\beta}$ - location of component CG within that component

$\bar{I}$ - without subscript, identity matrix

$\bar{I}$ - with subscript, inertia tensor

Abbreviations

CG - center of gravity

LDR - low drag ramjet

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